RESTRICTIONS ON THE LINEAR MODEL

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It is often stated that any set of "arbitrary, convenient restrictions" on the effects in the linear model may be used. The different sets of restrictions have certain properties; e.g., the resulting estimates are unbiased or are minimum variance unbiased; the particular restrictions used on the estimators hold for the parameters in the population; the restrictions are meaningful to the experimenter; etc. If it is desired to have unbiasedness, say, then it is not appropriate to use any "convenient restrictions" on the linear model but one must use restrictions from a selected set of restrictions.

The purpose of this article is to illustrate the effect of different sets of restrictions on the least squares estimates of effects for a number of simple examples. The more complex examples will not be considered here.

Some Examples

Example 1: The least squares estimates of effects for a two-way classification are [see Federer, Experimental Design , pages 127-129]:

$$\hat{\mu}=\bar{x},\ t_i=\hat{\tau}_i=\bar{x}_{i.}-\bar{x},\ \text{and}\ r_j=\hat{\rho}_j=\bar{x}_{.j}-\bar{x}$$

when the restrictions

$$\Sigma\hat{\tau}_i=0=\Sigma\hat{\rho}_j \qquad (1\text{-}1)$$

are used. Now, let's use the restrictions

$$\hat{\tau}_1=0=\Sigma\hat{\rho}_j \quad . \qquad (1\text{-}2)$$

From the normal equations on page 128 of Experimental Design , the following equations are obtained:

$$\hat{\mu}+n\sum_{i=2}^{v}\hat{\tau}_i \qquad =X_{..}\ ; \tag{1-3}$$

$$n\hat{\mu} \qquad =X_{1.}\ ; \tag{1-4}$$

$$n(\hat{\mu}+\hat{\tau}_i) \qquad =X_{i.} \tag{1-5}$$

$$v(\hat{\mu}+\hat{\rho}_j)+\sum_{i=2}^{v}\hat{\tau}_i \quad =X_{.j}\ . \tag{1-6}$$

From (1-4) we find $\hat{\mu}=\bar{x}_{1.}$ and from (1-5) we see that $\hat{\tau}_i$ (for $i=2,3,\cdots,v$)$=\bar{x}_{i.}-\bar{x}_{1.}$. Likewise, from (1-6), $\hat{\rho}_j=\bar{x}_{.j}-\bar{x}_{1.}-\frac{1}{v}\sum_{i=2}^{v}(\bar{x}_{i.}-\bar{x}_{1.})=\bar{x}_{.j}-\frac{1}{v}\sum_{i=1}^{v}\bar{x}_{i.}=\bar{x}_{.j}-\bar{x}$. Thus, the estimates of the $\hat{\rho}_j$ are unaltered when we use (1-2) instead of (1-1); the $\hat{\tau}_i$ using (1-2) are each decreased by an amount equal to $\bar{x}_{1.}-\bar{x}$, which is the estimate for $\hat{\tau}_1$ using (1-1); $\hat{\mu}$ using (1-2) is increased by an amount equal to $\hat{\tau}_1=\bar{x}_{1.}-\bar{x}$ over the estimate obtained using (1-1).

Likewise, we could use the restrictions

$$\hat{\rho}_1=0=\Sigma\hat{\tau}_i \tag{1-7}$$

and the results would be similar to those obtained using restrictions (1-2).

Regardless of the set of restrictions used, (1-1) or (1-2), the differences among the $\hat{\tau}_i$ and among the $\hat{\rho}_j$ will remain the same. For example, $\hat{\tau}_1-\hat{\tau}_2=\bar{x}_{1.}-\hat{\mu}-(\bar{x}_{2.}-\hat{\mu})=\bar{x}_{1.}-\bar{x}_{2.}$. Since the restrictions affect the value of $\hat{\mu}$ and since the coefficients of $\hat{\mu}$ add to zero in differences of the above sort, the experimenter may use the restrictions in (1-1) or (1-2) or some other convenient restriction if his interest lies only in estimating differences between parameters.

Example 2: If one uses restrictions of the form:

$$\hat{\tau}_1=0=\hat{\rho}_1 \quad (\text{or any } \hat{\tau}_i \text{ and } \hat{\rho}_j)\ , \tag{2-1}$$

the resulting normal equations are:

$$nv\hat{\mu}+n\sum_{i=2}^{v}\hat{\tau}_i+v\sum_{j=2}^{n}\hat{\rho}_j=X_{..} \quad ; \tag{2-2}$$

$$n\hat{\mu}+\sum_{j=2}^{n}\hat{\rho}_j=X_{1.} \quad ; \tag{2-3}$$

$$n(\hat{\mu}+\hat{\tau}_i)+\sum_{j=2}^{n}\hat{\rho}_j=X_{i.} \quad (\text{for } i=2,\cdots,v) \quad ; \tag{2-4}$$

$$v\hat{\mu}+\sum_{i=2}^{v}\hat{\tau}_i=X_{.1} \quad ; \tag{2-5}$$

$$v(\hat{\mu}+\hat{\rho}_j)+\sum_{i=2}^{v}\hat{\tau}_i=X_{.j} \quad (j=2,3,\cdots,n) \quad . \tag{2-6}$$

Subtracting (2-3) from (2-4), we obtain

$$\hat{\tau}_i=\bar{x}_{i.}-\bar{x}_{1.} \quad . \tag{2-7}$$

Likewise, subtracting (2-5) from (2-6), we obtain:

$$\hat{\rho}_j=\bar{x}_{.j}-\bar{x}_{.1} \quad . \tag{2-8}$$

Also,

$$\hat{\mu}=\bar{x}+(\bar{x}_{1.}-\bar{x})+(\bar{x}_{.1}-\bar{x})$$

$$=\bar{x}_{1.}+\bar{x}_{.1}-\bar{x} \tag{2-9}$$

Using (2-1), the $\hat{\tau}_i$ are the same as found by using (1-2) and the $\hat{\rho}_j$ are the same as found using (1-7). Thus, it is seen that differences between effects are unchanged by using the different restrictions given in equations (1-1), (1-2), (1-7) and (1-8).

Example 3: Now, let us use another type of restrictions, such as:

$$\hat{\tau}_1+\hat{\tau}_2=0=\Sigma\hat{\rho}_j \quad . \tag{3-1}$$

The normal equations become:

$$nv\hat{\mu}+n\sum_{i=3}^{v}\hat{\tau}_i \quad =X_{..} \quad ; \tag{3-2}$$

$$n(\hat{\mu}+\hat{\tau}_1) \quad =X_{1.} \quad ; \tag{3-3}$$

$$n(\hat{\mu}+\hat{\tau}_2) \quad =X_{2.} \quad ; \tag{3-4}$$

$$n(\hat{\mu}+\hat{\tau}_i) \quad =X_{i.} \quad (i=3,4,\cdots,v) \quad ; \tag{3-5}$$

$$v(\hat{\mu}+\hat{\rho}_j)+\sum_{i=3}^{v}\hat{\tau}_i \quad =X_{.j} \quad . \tag{3-6}$$

From the above,

$$\hat{\mu}=(\bar{x}_{1.}+\bar{x}_{2.})/2 \quad ; \tag{3-7}$$

$$\hat{\tau}_i=\bar{x}_{i.}-(\bar{x}_{1.}+\bar{x}_{2.})/2 \quad ; \tag{3-8}$$

$$\hat{\rho}_j=\bar{x}_{.j}-\bar{x} \quad . \tag{3-9}$$

In the above, we note that the estimates $\hat{\rho}_j$ remained unchanged using (3-1) instead of (1-1). The estimates $\hat{\tau}_i(i=3,\cdots,v)=\bar{x}_{i.}-\bar{x}-\dfrac{(\bar{x}_{1.}-\bar{x}+\bar{x}_{2.}-\bar{x})}{2}$ from (3-8) need to be increased by an amount $(\bar{x}_{1.}-\bar{x}+\bar{x}_{2.}-\bar{x})/2$ to obtain the $\hat{\tau}_i$ using restriction (1-1); differences among the $\hat{\tau}_i$ remain unchanged.

Example 4: Consider now the completely randomized design described in Federer's Experimental Design on pages 104-5, which consists of v treatments with r_i replicates on the ith treatment. Instead of the restriction $\sum_{i=1}^{v}\hat{\tau}_i=0$, use the restriction $\sum_{i=1}^{v} r_i\hat{\tau}_i=0$. Then, $\hat{\mu}=\bar{x}$, and

$$\hat{\tau}_i=\bar{x}_{i.}-\bar{x} \quad . \tag{4-1}$$

The differences among the $\hat{\tau}_i$ will remain unchanged regardless of the linear restriction used on the estimates.

Example 5: Assuming that the equation expressing the yield of the hth observation in the ijth subclass for a two-way classification with n_{ij} observations per subclass, is of the form:

$$X_{ijh}=\mu+\tau_i+\rho_j+\epsilon_{ijh} \quad , \tag{5-1}$$

the normal equations are:

$$n_{..}\hat{\mu}+\sum_{i=1}^{v} n_{i.}\hat{\tau}_i+\sum_{j=1}^{r} n_{.j}\hat{\rho}_j=X_{...} = \text{grand total;} \tag{5-2}$$

$$n_{i.}(\hat{\mu}+\hat{\tau}_i)+\sum_{j=1}^{r} n_{ij}\hat{\rho}_j=X_{i..} = \text{ith treatment total;} \tag{5-3}$$

$$n_{.j}(\hat{\mu}+\hat{\rho}_j)+\sum_{i=1}^{v} n_{ij}\hat{\tau}_i=X_{.j.} = \text{jth block total,} \tag{5-4}$$

where $n_{i.}=\sum_{j=1}^{r} n_{ij}$; $n_{.j}=\sum_{i=1}^{v} n_{ij}$; and $n_{..}=\sum_{i=1}^{v}\sum_{j=1}^{r} n_{ij}$.

Solving for $\hat{\mu}+\hat{\rho}_j$ from (5-4) and substituting in (5-3), v equations in the $\hat{\tau}_i$ are obtained. The kth equation in this set is

$$\hat{\tau}_k\left(n_{k.}-\sum_{j=1}^{r}\frac{n_{kj}^2}{n_{.j}}\right)-\sum_{j=1}^{r}\sum_{\substack{i=1\\ \neq k}}^{v}\frac{n_{kj}n_{ij}}{n_{.j}}\hat{\tau}_i=X_{k..}-\sum_{j=1}^{r} n_{kj}\bar{x}_{.j.}=Q_{k.} \quad , \tag{5-5}$$

where $\bar{x}_{.j.}$ = arithmetic mean for the jth block.

Likewise, substitution of the $(\hat{\mu}+\hat{\tau}_i)$ values from (5-3) in (5-4) results in r equations in the $\hat{\rho}_j$. The gth equation of this set is:

$$\hat{\rho}_g(n_{\cdot g}-\sum_{i=1}^{v}\frac{n_{ig}^2}{n_{i\cdot}})-\sum_{i=1}^{v}\sum_{\substack{j=1\\ \neq g}}^{r}\frac{n_{ij}n_{ig}}{n_{i\cdot}}\hat{\rho}_j=X_{\cdot g\cdot}-\sum_{i=1}^{v}n_{ig}\bar{x}_{i\cdot\cdot}=Q_{\cdot g}\ , \tag{5-6}$$

where $\bar{x}_{i\cdot\cdot}$ = arithmetic mean for the ith treatment. The addition of restrictions such as

$$\Sigma\hat{\tau}_i=0=\Sigma\hat{\rho}_j \tag{5-7}$$

or

$$\hat{\tau}_1=0=\hat{\rho}_1\ , \tag{5-8}$$

results in unique solutions for the $\hat{\tau}_i$ and the $\hat{\rho}_j$, and the differences among the $\hat{\tau}_i$ and among the $\hat{\rho}_j$ will be the same using restrictions (5-7) or (5-8).

If restrictions of the form,

$$\Sigma n_{i\cdot}\hat{\tau}_i=0=\Sigma n_{\cdot j}\hat{\rho}_j\ , \tag{5-9}$$

are used, the resulting differences among the $\hat{\tau}_i$ and among the $\hat{\rho}_j$ will be the same as found using restrictions (5-7). For example, consider the case where v=2; the two equations involving the $\hat{\tau}_i$ are:

$$(n_{1\cdot}-\sum_{j=1}^{r}\frac{n_{1j}^2}{n_{\cdot j}})\hat{\tau}_1-\sum_{j=1}^{r}\frac{n_{1j}n_{2j}}{n_{\cdot j}}\hat{\tau}_2=X_{1\cdot\cdot}-\Sigma n_{1j}\bar{x}_{\cdot j\cdot}=Q_{1\cdot} \tag{5-10}$$

and

$$-\Sigma\frac{n_{1j}n_{2j}}{n_{\cdot j}}\hat{\tau}_1+(n_{2\cdot}-\sum_{j=1}^{r}\frac{n_{2j}^2}{n_{\cdot j}})\hat{\tau}_2=X_{2\cdot\cdot}-\Sigma n_{2j}\bar{x}_{\cdot j\cdot}=Q_{2\cdot} \tag{5-11}$$

Using the restrictions in (5-7)

$$\hat{\tau}_1=Q_1/(n_{1\cdot}-\Sigma\frac{n_{1j}^2}{n_{\cdot j}}+\sum_j\frac{n_{1j}n_{2j}}{n_{\cdot j}}) \tag{5-12}$$

and

$$\hat{\tau}_2=-\hat{\tau}_1=Q_2/(n_{2.}-\Sigma\frac{n_{2j}^2}{n_{.j}}+\Sigma\frac{n_{1j}n_{2j}}{n_{.j}}) \ , \qquad (5\text{-}13)$$

where $n_{1.}-\Sigma\ n_{1j}^2/n_{.j}=n_{2.}-\Sigma\ n_{2j}^2/n_{.j}=\Sigma\ n_{1j}n_{2j}/n_{.j}$ and where $Q_1=-Q_2$. Using restrictions (5-9) the estimates are:

$$\hat{\tau}_1=Q_1/(n_{1.}-\Sigma n_{1j}^2/n_{.j}+\frac{n_{1.}}{n_{2.}}\Sigma n_{1j}n_{2j}/n_{.j}) \qquad (5\text{-}14)$$

and

$$\hat{\tau}_2=Q_2/(n_{2.}-\Sigma n_{2j}^2/n_{.j}+\frac{n_{2.}}{n_{1.}}\Sigma n_{1j}n_{2j}/n_{.j}) \qquad (5\text{-}15)$$

From (5-12) and (5-13) we find

$$\hat{\tau}_1-\hat{\tau}_2=\frac{Q_1-Q_2=2Q_1}{n_{1.}-\Sigma n_{1j}^2/n_{.j}+\Sigma n_{1j}n_{2j}/n_{.j}}=\frac{Q_1}{n_{1.}-\Sigma n_{1j}^2/n_{.j}} \ . \qquad (5\text{-}16)$$

From (5-14) and (5-15) we find

$$\hat{\tau}_1-\hat{\tau}_2=\frac{Q_1}{K(1+n_{1.}/n_{2.})}-\frac{Q_2}{K(1+n_{2.}/n_{1.})}=\frac{Q_1}{K} \ , \qquad (5\text{-}17)$$

where $K=\Sigma n_{1j}n_{2j}/n_{.j}=n_{1.}-\Sigma n_{1j}^2/n_{.j}=n_{2.}-\Sigma n_{2j}^2/n_{.j}$ and where $Q_1=-Q_2$. The differences between the two sets of estimates are the same. Hence, it does not matter what linear restriction is imposed on the $\hat{\tau}_i$, the differences between the estimates will be identical. Also, use of the restrictions $\Sigma\hat{\tau}_i=\Sigma n_{.j}\hat{\rho}_j=0$ result in the same differences between estimated effects as obtained above.

Example 6: The least squares estimates for a two-way classification with k items per subclass and with interaction are given on pages 131-132 of Federer's Experimental Design as:

$$\hat{\mu} = \bar{\bar{x}} \quad , \tag{6-1}$$

$$\hat{\tau}_i = \bar{x}_{i..} - \bar{\bar{x}} \quad , \tag{6-2}$$

$$\hat{\rho}_j = \bar{x}_{.j.} - \bar{\bar{x}} \quad , \tag{6-3}$$

and

$$\widehat{\rho\tau}_{ij} = \bar{x}_{ij.} - \bar{x}_{i..} - \bar{x}_{.j.} + \bar{\bar{x}} \quad . \tag{6-4}$$

The restrictions used to obtain the above estimates are:

$$\sum_{i=1}^{v} \hat{\tau}_i = 0 = \sum_{j=1}^{r} \hat{\rho}_j \quad ; \tag{6-5}$$

$$\sum_{i=1}^{v} \widehat{\rho\tau}_{ij} = 0 = \sum_{j=1}^{r} \widehat{\rho\tau}_{ij} \quad . \tag{6-6}$$

Suppose we use the following instead of (6-5)

$$\tau_1 = 0 = \Sigma\hat{\rho}_j \quad . \tag{6-7}$$

The least squares estimates of the effects become:

$$\hat{\mu} = \bar{x}_{1..} \quad ; \tag{6-8}$$

$$\hat{\tau}_i = \bar{x}_{i..} - \bar{x}_{1..} \quad ; \tag{6-9}$$

$$\hat{\rho}_j = \bar{x}_{.j.} - \bar{\bar{x}} \quad ; \tag{6-10}$$

$$\widehat{\rho\tau}_{ij} = \bar{x}_{ij.} - \bar{x}_{.j.} - \bar{x}_{i..} + \bar{\bar{x}} \quad . \tag{6-11}$$

Thus, the $\hat{\rho}_j$ and $\widehat{\rho\tau}_{ij}$ estimates and the differences among the $\hat{\tau}_i$ are unaltered if we use (6-6) and either (6-5) or (6-7).

Consider now that the following restrictions are used instead of (6-6):

$$\widehat{\rho\tau}_{ir} \text{ (for every } i) = 0 = \widehat{\rho t}_{vj} \text{ (for every } j) \text{ .} \tag{6-12}$$

The normal equations, using (6-5) and (6-12), now become:

$$rkv\,\hat{\mu}+k\sum_{i=1}^{v-1}\sum_{j=1}^{r-1}\widehat{\rho\tau}_{ij}=X_{\cdot\cdot\cdot} = \text{grand total ;} \tag{6-13}$$

$$rk(\hat{\mu}+\hat{\tau}_i)+k\sum_{j=1}^{r-1}\widehat{\rho\tau}_{ij}=X_{i\cdot\cdot} = \text{ith treatment total ;} \tag{6-14}$$

$$vk(\hat{\mu}+\hat{\rho}_j)+k\sum_{i=1}^{v-1}\widehat{\rho\tau}_{ij}=X_{\cdot j\cdot} = \text{jth replicate total ;} \tag{6-15}$$

$$k(\hat{\mu}+\hat{\tau}_i+\hat{\rho}_j+\widehat{\rho\tau}_{ij})=X_{ij\cdot} = \text{total of ijth cell .} \tag{6-16}$$

From the above equations and equations (6-5) and (6-12), the estimates of the effects are found to be:

$$\hat{\mu}=\frac{1}{vk}\sum_{i=1}^{v}X_{ir\cdot}+\frac{1}{rk}\sum_{j=1}^{r}X_{vj\cdot}-\frac{1}{k}X_{vr\cdot}=\bar{x}_{\cdot r\cdot}+\bar{x}_{v\cdot\cdot}-\bar{x}_{vr\cdot} \quad ; \tag{6-17}$$

$$\hat{\rho}_j=\bar{x}_{vj\cdot}-\bar{x}_{v\cdot\cdot} \quad ; \tag{6-18}$$

$$\hat{\tau}_i=\bar{x}_{ir\cdot}-\bar{x}_{\cdot r\cdot} \quad ; \tag{6-19}$$

$$\widehat{\rho\tau}_{ij}=\bar{x}_{ij\cdot}-\bar{x}_{vj\cdot}-\bar{x}_{ir\cdot}+\bar{x}_{vr\cdot} \quad . \tag{6-20}$$

The estimates of the effects and the differences between estimates are <u>not</u> the same in the two sets of equations, (6-17) to (6-20) and (6-8) to (6-11). Here then, we are not allowed to use any "convenient restriction" if we wish estimated differences between effects to remain the same as those obtained using the restrictions in (6-5) and (6-6).

Definitions and Restrictions on the Parameters

Given that

$$X_{ijh}=\mu_{ij}+\epsilon_{ijh} \tag{1}$$

and that we have a random sample from the ijth population we can estimate the population mean μ_{ij} (i.e., μ_{ij} is estimable) and obtain an estimate of the variance of the mean. The estimator of μ_{ij} is $\bar{x}_{ij.}$, the subclass mean.

Now suppose that we set

$$X_{ijh}=\mu+\tau_i+\rho_j+\rho\tau_{ij}+\epsilon_{ijh} \tag{2}$$

and say that we are in the fixed effects situation, that μ = an effect common to every member in the sample, τ_i = an effect common to the ith member of the first classification, say treatments, and ρ_j = an effect common to the jth member of the second classification, say replicates, and $\rho\tau_{ij}$ = an effect common to the ith treatment in the jth block, and the ϵ_{ijh} are independently distributed with mean zero and constant variance. The μ, τ_i, ρ_j, and $\rho\tau_{ij}$ are not estimable. If we further state that*

$$\Sigma\rho_j=\Sigma\tau_i=\sum_i\rho\tau_{ij}=\sum_j\rho\tau_{ij}=0 \ , \tag{3}$$

it is then possible to estimate μ, differences among the τ_i, differences among the ρ_j, and differences among the $\rho\tau_{ij}$. Note that the τ_i, ρ_j, and $\rho\tau_{ij}$ are not estimable.

Going one step further we could use the following definitions of the effects in equation (2) defining μ_{ij} as in (1):

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*N.B. τ_1=100 and $\sum_{i=1}^{v}\tau_i$=-100 would satisfy the condition that $\Sigma\tau_i$= zero.

$$\mu = \frac{1}{rv} \sum_{i=1}^{v} \sum_{j=1}^{r} \mu_{ij} = \mu_{..} \quad ; \tag{4}$$

$$\tau_i = \frac{1}{r} \sum_j \mu_{ij} - \mu_{..} = \mu_{i.} - \mu_{..} \quad ; \tag{5}$$

$$\rho_j = \frac{1}{v} \sum_i \mu_{ij} - \mu_{..} = \mu_{.j} - \mu_{..} \quad ; \tag{6}$$

$$\rho\tau_{ij} = \mu_{ij} - \mu_{i.} - \mu_{.j} + \mu_{..} \quad . \tag{7}$$

These definitions imply the equalities in (3).

From a theorem on minimum variance unbiased estimators (e.g., see H. B. Mann, "Analysis and design of experiments," chapters 4-5, 1949) we simply replace the parameter, μ_{ij}, by the estimator, $\bar{x}_{ij.}$, to obtain a minimum variance unbiased estimator of a linear function of the parameters μ_{ij}. Thus,

$$\hat{\tau}_i = \frac{1}{r} \sum_j \bar{x}_{ij.} - \bar{x} = \bar{x}_{i..} - \bar{x} \quad ;$$

$$\hat{\rho}_j = \frac{1}{v} \sum_i \bar{x}_{ij.} - \bar{x} = \bar{x}_{.j.} - \bar{x} \quad ;$$

$$\widehat{\rho\tau}_{ij} = \bar{x}_{ij.} - \bar{x}_{i..} - \bar{x}_{.j.} + \bar{x} \quad .$$

The τ_i, ρ_j, and $\rho\tau_{ij}$ are estimable using definitions (5), (6), and (7), but they were not estimable using only the equalities in (3).

Confining our attention to the fixed effects case and to estimation, it would be entirely reasonable to insist on the definitions given by equations (4) to (7). For hypothesis testing only definition (3) would suffice, but definitions (4) to (7) do not rule out hypothesis testing. Hence, the use of the latter definitions has considerable appeal in that the actual population description is made precise and the effects are all estimable.

Restrictions on the Estimators

From the normal equations for the fixed effects case (either from least squares or maximum likelihood) for a two-way classification with interaction, we find that

$$\hat{\mu}+\hat{\tau}_i=\bar{x}_{i..}-\frac{1}{rk}\left\{\sum_j(\hat{\rho}_j+\widehat{\rho\tau}_{ij}\right\} \quad . \tag{8}$$

If we use the restriction

$$\sum_j(\hat{\rho}_j+\widehat{\rho\tau}_{ij})=0 \ , \tag{9}$$

the estimate of $\mu+\tau_i$ is $\widehat{\mu+\tau}_i=\bar{x}_{i..}$; no restrictions have been placed on the $\hat{\tau}_i$. But, in order to estimate the ρ_j and $\rho\tau_{ij}$, some restriction must be placed on the $\hat{\tau}_i$ and an additional restriction must be placed on the $\widehat{\rho\tau}_{ij}$. Suppose that we use equations (6-6) and (6-7). The estimators of the effects are given in (6-8) to (6-11). Now if

$$E(\bar{x}_{i..}/\text{ith sample treatment = Ith population treatment})=\mu+\tau_i \ , \tag{10}$$

$$E(\bar{x}_{.j.}/\text{jth sample replicate = Jth population replicate})=\mu+\rho_j \ , \tag{11}$$

and

$$E(\bar{x}_{ij.}/\text{ith treatment in jth replicate in sample = Ith treatment in Jth replicate in the population})=\mu+\tau_i+\rho_j \ , \tag{12}$$

the differences between effects are estimable, e.g., using (10) to (12),

$$E(\bar{x}_{i..}-\bar{x}_{i'..})=\tau_i-\tau_{i'} \ , \tag{13}$$

$$E(\bar{x}_{.j.}-x_{.j'.})=\rho_j-\rho_{j'} \ , \tag{14}$$

and

$$E(\widehat{\rho\tau}_{ij}-\widehat{\rho\tau}_{i'j'})=\rho\tau_{ij}-\rho\tau_{i'j'} \quad , \tag{15}$$

where $\widehat{\rho\tau}_{ij}$ is obtained from (6-11).

If restrictions (6-5) and (6-12) are used the estimators are given in (6-17) to (6-20); the differences among effects are also estimable, thus

$$E(\hat{\tau}_i-\hat{\tau}_{i'}=\bar{x}_{ir.}-\bar{x}_{i'r.})=\tau_i-\tau_{i'} \quad , \tag{16}$$

$$E(\hat{\rho}_j-\hat{\rho}_{j'}=\bar{x}_{vj.}-\bar{x}_{vj'.})=\rho_j-\rho_{j'} \quad , \tag{17}$$

$$E(\widehat{\rho\tau}_{ij}-\widehat{\rho\tau}_{i'j'}=\bar{x}_{ij.}-\bar{x}_{vj.}-\bar{x}_{ir.}-\bar{x}_{i'j'}+\bar{x}_{vj'.}+\bar{x}_{i'r.})=\rho\tau_{ij}-\rho\tau_{i'j'} \quad . \tag{18}$$

From the above it might appear that the estimators from (6-1) to (6-4) and from (6-17) to (6-20) are equivalent. They both yield unbiased estimates of differences between the effects. However, the estimators in (6-17) to (6-20) have expectations as follows:

$$E(\hat{\mu}=\bar{x}_{.r.}+\bar{x}_{v..}-\bar{x}_{rv.})=\mu \quad ; \tag{19}$$

$$E(\hat{\rho}_j=\bar{x}_{vj.}-\bar{x}_{v..})=\rho_j-\rho_r+\tau_v \quad ; \tag{20}$$

$$E(\hat{\tau}_i=\bar{x}_{ir.}-\bar{x}_{.r.})=\tau_i-\tau_v+\rho_r \quad ; \tag{21}$$

$$E(\widehat{\rho\tau}_{ij}=\bar{x}_{ij.}-\bar{x}_{vj.}-\bar{x}_{ir.}+\bar{x}_{vr.})=\rho\tau_{ij} \quad . \tag{22}$$

The $\hat{\rho}_j$ and $\hat{\tau}_i$ estimates are biased. Furthermore, none of the estimates in (6-17) to (6-20) are minimum variance unbiased estimators. To obtain the minimum variance unbiased estimators ((6-1) to (6-4)) we simply substitute the sample value $\bar{x}_{ij.}$ for μ_{ij} in (4) to (7).

Discussion of Examples

From the foregoing it has been shown that any "convenient and arbitrary" restriction on the estimates cannot be used if properties like minimum variance and unbiasedness are to be retained. If the restrictions applicable to parameters are used on the estimated effects, then the minimum variance unbiased property is retained.

For the situations encountered in examples 1 to 5 the differences between estimated effects are minimum variance unbiased estimates of the differences between the corresponding parameters, regardless of the linear restrictions on the estimated effects. For classifications without interactions, the use of "arbitrary convenient" linear restrictions on the estimators appears justifiable when estimated differences between effects are the only items of concern. However, as soon as interactions are involved in two-way and higher-way classifications, it appears to be necessary to use the same restrictions on the estimators as are applicable to the parameters.